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Pitch Pointing Flight Control System Design in the Frequency Domain

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Introduction

ADVANCED aircraft provide some significant modes for bombing and air-to-air combat. One of the main control objectives for these advanced modes is the decoupling of the multivariable aircraft system. For the longitudinal dynamics of an aircraft, one such decoupled mode is pitch pointing, which is characterized by decoupling the pitch attitude and flight-path angle.

Sobel and Shapiro[1] have proposed a design methodology that uses eigenstructure assignment to decouple the system, and have obtained the desired properties of the closed-loop feedback system in state-space. In this Note, an alternative frequency domain design that decouples the multivariable system by using an H^∞-optimization technique is proposed, and a stable minimum-phase weighting function to meet the desired damping ratio and the rise time is constructed. Using the H^∞-optimization technique, the multivariable problem is treated exactly as the scalar problem in the pitch pointing control design, and a diagonal closed-loop transfer function matrix of the multivariable system is obtained. The result of the pitch pointing controller design shows the perfect system decoupling, which achieves an attitude command tracking but without causing any flight path change, and possesses desired tracking properties.

H^∞-Optimization Design Methodology

Consider the dynamical equation of the aircraft plant in the pitch pointing control system, which is described in Fig. 1 by

$$\dot{x} = Ax + Bu \tag{1}$$

$$y = Dx \tag{2}$$

The plant is modeled by the loop transfer function matrix

$$P = D(sI - A)^{-1} B \tag{3}$$

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If $\det(sI - A)$ has right-half-plane (RHP) zeros, then this kind of aircraft without augmentation is unstable. For such an aircraft, does there exist a controller $C(s)$ such that the closed-loop multivariable system is decoupled, and possesses an optimal asymptotic tracking property under some design criterion? In this Note, we propose an H^∞-optimization design criterion as follows:

Theorem: Consider the $n \times n$ multivariable system given in Fig. 1. Let the input r be a unit step function, T_o be the closed-loop transfer function matrix, and W be a stable minimum-phase diagonal weighting function matrix. If $\det(sI - A)$ in Eq. (3) has RHP zeros, then there exists a diagonal optimal all-pass function matrix $\tilde{G}_o \in H^\infty$, the bounded and analytic function matrix on RHP, such that

$$\|\tilde{G}_o(s)\|_\infty = \inf_{C(s) \in H^\infty} \|W(s)T_o(s)\|_\infty = \tilde{k} \tag{4}$$

and the overall transfer function matrix in steady state would be

$$T(0) = K\tilde{T}_o(0) = KW^{-1}(0)\tilde{G}_o(0) = I \tag{5}$$

where W is viewed as a design parameter selected to reflect the desired properties of the optimal transfer function matrix $\tilde{T}_o$ and make the infimum $\tilde{k}$ less than one. (The sufficient condition[3] $\inf_{C \in H^\infty} \|WT_o\|_\infty < 1$ assures the closed-loop system optimal robust for all multiplicative plant perturbation $\Delta_m(s)$ with $\bar{\sigma}[\Delta_m(j\omega)] < \bar{\sigma}[W(j\omega)] \forall \omega \geq 0$, where $\bar{\sigma}[\cdot]$ denotes the maximum singular value of the matrix $[\cdot]$.) K is a constant-gain matrix chosen to make the overall system asymptotic tracking; $\| \; \|_\infty$ is the maximum modulus norm, and $\tilde{T}_o$ denotes the optimal closed-loop transfer function matrix.

This theorem implies that if the diagonal all-pass function matrix $\tilde{G}_o$ is attained, then the closed-loop transfer function matrix $\tilde{T}_o(s) = W^{-1}(s)\tilde{G}_o(s)$ can be decoupled, and the optimal asymptotic tracking property as Eq. (5) will be obtained.

Proof: Let $P = Nd^{-1}$, where d is the Blaschke product of the RHP zeros of $\det(sI - A)$. Select $X_i, Y_i \in H^\infty$, $i = 1,2$, such that

$$X_1N + dY_1 = I, \quad NX_2 + dY_2 = I \text{ (Bezout Identity)} \tag{6}$$

Then T_o can be parametrized as[3]

$$T_o = I - dY_2 + dNR \tag{7}$$

and the problem in Eq. (4) can be reduced to find

$$\inf_{C \in H^\infty} \|WT_o\|_\infty \tag{8}$$

$$= \inf_{R \in H^\infty} \|W(NX_2 + dNR)\|_\infty \quad \text{[by Eq. (6) and (7)]} \tag{9}$$

$$= \inf_{R \in H^\infty} \|W(N)_i(N)_o(X_2 + dR)\|_\infty$$

[by inner-outer factorization $N = (N)_i(N)_o$] (10)

$$= \inf_{R' \in H^\infty} \|W(N)_oX_2 + dR'\|_\infty$$

[assume that $(N)_i$ is diagonal in the following example $(N)_i = I$] (11)

$$= \inf_{R' \in H^\infty} \|F + dR'\|_\infty \tag{12}$$

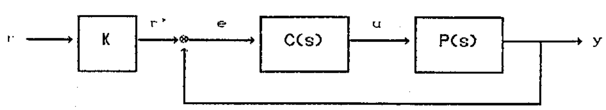

Fig. 1 Typical control system.

Assume that s_i is the zero of $d(s)$, then we have

$$NX_2(s_i) + dY_2(s_i) = (N)_i(N)_o X_2(s_i) = I \tag{13}$$

and

$$F(s_i) = W(N)_o X_2(s_i) = W[(N)_i]^{-1}(s_i) \tag{14}$$

will be a diagonal matrix. Using the Nevanlinna-Pick algorithm,[2] we can find a diagonal optimal function matrix $\tilde{G}(s)$ such that

$$\|\tilde{G}\|_\infty = \inf_{R' \in H^\infty} \|F + dR'\|_\infty = \tilde{k} \tag{15}$$

and the corresponding matrix $\tilde{G}_o$ for the original problem in Eq. (4) is

$$\tilde{G}_o = (N)_i \tilde{G} \tag{16}$$

Therefore, we will obtain an optimal decoupled system transfer function matrix

$$\tilde{T}_o(s) = W^{-1}(s)\tilde{G}_o(s) \tag{17}$$

and adjust the constant-gain matrix K such that

$$T(0) = KW^{-1}(0)\tilde{G}_o(0) = I \tag{18}$$

an optimal asymptotic tracking will then be attained. *Q.E.D.*

Construction of a Weighting Function $W(s)$

Assume

$$W_i(s) = \frac{(bs + c)(s^2 + ds + e)}{s + a}, \qquad i = 1, \cdots, n \tag{19}$$

and for the given damping ratio ξ, natural frequency ω_n, and the gain margin GM at crossover frequency ω_c, using Eqs. (17) and (19) we have the following constraint equations:

$$d = 2\xi\omega_n \tag{20a}$$

$$e = \omega_n^2 \tag{20b}$$

$$\tilde{G}_o(j\omega_c) = W(j\omega_c)\tilde{T}_o(j\omega_c) \tag{20c}$$

The above equations and Eq. (4) are sufficient to solve for all coefficients.

Example

Consider the dynamic model of an advanced aircraft that is described by

$$\dot{x} = Ax + Bu \tag{21a}$$

$$y = Dx \tag{21b}$$

where

$$x = [\theta \; q \; \alpha \; \delta_e \; \delta_f]^T$$

$$y = [\theta \; r]^T, \qquad u = [\delta_{e_c} \; \delta_{f_c}]^T$$

q = pitch rate
r = flight-path angle
θ = pitch attitude
α = angle of attack
δ_e = elevator deflection
δ_{e_c} = elevator deflection command
δ_f = flaperon deflection
δ_{f_c} = flaperon deflection command

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -0.86939 & 43.223 & -17.251 & -1.5766 \\ 0.004 & 0.99335 & -1.3411 & 0.16897 & 0.025 \\ 0 & 0 & 0 & -20 & 0 \\ 0 & 0 & 0 & 0 & -20 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 20 & 0 \\ 0 & 20 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \end{bmatrix}$$

By Eq. (3), we have

$$P = \frac{1}{-(s + 0.00413873)(s + 20)(s + 7.66029)(s - 5.4539)} \begin{bmatrix} 345.02(s + 0.91774) & 31.532(s + 0.65572) \\ 3.38s^2 + 5.232s + 315.258 & 0.5s^2 + 0.6444s + 20.55 \end{bmatrix} = Nd^{-1}$$

where $d = [(s - 5.4539)/(s + 5.4539)]$, which is a Blaschke product form of the unstable pole of $P(s)$.

Because $(N)_i = I$ and $T_o = I - dY_2 + d(N)_o R = I + d[-Y_2 + (N)_o R]$, Eq. (8) can be reduced to find

$$\inf_{R' \in H^\infty} \|W + dR'\|_\infty \tag{22}$$

Assume the given damping ratio and natural frequency to be

$$\xi = 0.9, \qquad \omega_n = 3.6$$

Then we have

$$d = 6.48, \qquad e = 12.96$$

Because $d(s)$ only has one zero ($s_1 = 5.4539$), and assuming

$$W_i = \frac{(bs + c)(s^2 + ds + e)}{s + a}, \qquad \forall_i, \; i = 1,2 \tag{23}$$

the optimal diagonal all-pass function matrix $\tilde{G}_o(s)$ will be

$$\tilde{G}_o(s) = \tilde{k}I_2 = \bar{\sigma}[W(5.4539)]I_2 \tag{24}$$

For the gain margin $GM \geq 8\ dB$ at $\omega_c = 10$ rad/s, by Eq. (20) we have

$$\begin{aligned} \tilde{G}_o(j10) &= \bar{\sigma}[W(5.4539)]I_2 \\ &= W(j10)\,[1 - 1/(1 - GM^{-1})]I_2 \end{aligned} \tag{25}$$

If we choose $\tilde{k} = \bar{\sigma}[W(5.4539)] = 0.8$, then the weighting function $W(s)$ can be constructed approximately as

$$W(s) = \frac{(s^2 + 6.48s + 12.96)}{s + 31.7634} \begin{bmatrix} 0.00004s^2 + 0.030082s + 0.216224 & 0 \\ 0 & 0.030082s + 0.217414 \end{bmatrix} \tag{26}$$

Adjust the constant-gain K, such that

$$KW^{-1}(0)\tilde{G}_o(0) = I_2 \tag{27}$$

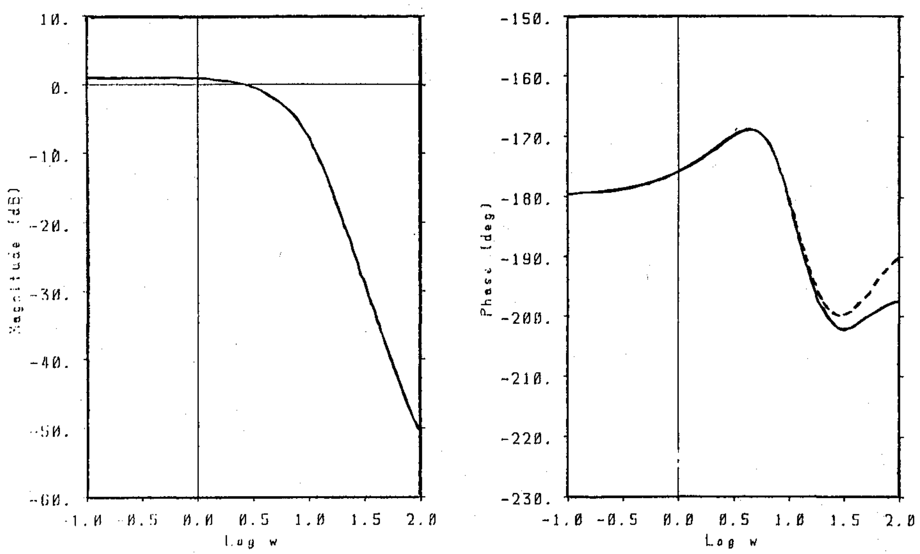

Fig. 2 Bode diagram of open-loop transfer function $P(s)\tilde{C}(s)$: a) magnitude plot, and b) phase plot.

Then K can be calculated as

$$K = \begin{bmatrix} 0.1102788 & 0 \\ 0 & 0.110889 \end{bmatrix} \tag{28}$$

Hence, the closed-loop transfer function $\tilde{T}_o(s)$ will be

$$\tilde{T}_o(s) = W^{-1}(s)\tilde{G}_o(s)$$
$$= \begin{bmatrix} \dfrac{s + 31.7634}{(s^2 + 6.48s + 12.96)(0.00005s^2 + 0.037603s + 0.27028)} & 0 \\ 0 & \dfrac{s + 31.7634}{(s^2 + 6.48s + 12.96)(0.037603s + 0.271767)} \end{bmatrix} \tag{29}$$

and the optimal controller $\tilde{C}(s)$ can be calculated as

$$\tilde{C}(s) = \frac{-(s+31.7634)(s+0.00414)(s+20)(s+7.6603)}{65.95076s^3+145.7822s^2-2754.753s-11.44505} \begin{bmatrix} \dfrac{0.5s^2+0.6444s+20.55}{0.00005s^3+0.0382s^2+0.7220s+5.18154} & \dfrac{-31.532(s+0.65572)}{0.0376s^2+0.7205s+5.178} \\ \dfrac{-(3.38s^2+5.23s+315.258)}{0.00005s^3+0.0382s^2+0.7220s+5.18154} & \dfrac{345.02(s+0.91774)}{0.0376s^2+0.7205s+5.178} \end{bmatrix} \tag{30}$$

The open-loop transfer function

$$P(s)\tilde{C}(s) = \begin{bmatrix} \dfrac{s+31.7634}{0.00005s^4+0.03793s^3+0.5146s^2+1.23875s-28.261} & 0 \\ 0 & \dfrac{s+31.7634}{0.0376s^3+0.5154s^2+1.2486s-28.2409} \end{bmatrix} \tag{31}$$

and its frequency response is shown in Fig. 2. From the Bode diagram of Fig. 2. and Eq. (4), we know that the tolerable plant perturbation Δ_m is small because the sufficient condition of system robust and $\bar{\sigma}[\Delta_m(j\omega)] < \bar{\sigma}[W(j\omega)]$ must be held.

Discussion

1) Proper controller problem: Because the solution of the H^∞-optimization is an all-pass function, in order to construct a proper (realizable) controller, we choose an improper weighting function as shown in Eq. (23).

2) Reduced-order suboptimal controller design: If $\tilde{T}_o(s)$ can be obtained from Eq. (17), then

$$\tilde{C}(s) = P^{-1}(s)[I - \tilde{T}_o(s)]^{-1}\tilde{T}_o(s) \tag{32}$$

The order of the optimal controller $\tilde{C}(s)$, which depends on the order of $P(s)$ and $\tilde{T}_o$, may be very high. Because of the limitation of the computation speed, it is desired to design a reduced-order suboptimal controller, and evaluate the degraded amount of the closed-loop system robustness and decoupling. This is our area of future research.

3) It is interesting to compare the solution of the H^∞-optimization in frequency domain design with the solution of the eigenstructure assignment in state-space design.

Theoretically, although we have a simple gain in the eigenstructure design, it is difficult to decouple perfectly a multivariable system. However, we can obtain a complete decoupled system in the H^∞-optimization design as long as the diagonal optimal solution can be attained. Certainly, this design need pay the penalty of the higher-order controller.

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Optimal Transfer from Collinear Libration Points with Limited Rotation Speed

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I. Introduction

THE papers[2-5] have dealt with the problem of reaching the collinear and equilateral libration points of the Earth–Moon system of a space vehicle that is placed in Earth or Moon orbit.

It has been seen, using the impulse method, that small variations in the initial velocity result in important deviations from the libration points. This disadvantage can be eliminated by the continuous use of a low thrust that leads to a large class of problems concerning several requirements of optimizing certain quantities, such as the fuel consumption and the period of transfer for evolutions of short periods.

The present paper studies the transfer in minimum time of a mass particle initially at rest at the libration point, using a small thrust with constant magnitude and bounded angular speed to achieve a prescribed final velocity. The restrictions divide the optimal trajectory into three arcs. On this trajectory, the rotation velocity of the direction of the propulsion has the extremal value or corresponds to Lawden's tangent law. The matching of the arcs together with transversality conditions and final conditions determines the constants of integration and the evolution time. The equations of motion are integrated from the assigned initial conditions. The final values on one arc are taken as initial falues for the next arc. Commutation times t_1 and t_2 of the control, the integration constants of the adjoint system, and the total time are determined from the continuity requirement at the end of the first arc and from the transversality and initial conditions.

II. Equations of Motion

In the phase space $X = \{x_i\}$ $(i = 1, \dots, 5)$, the equations of motion in a rotating system of axes with the origin at the collinear libration points of a vehicle acted upon by a propulsion force of small magnitude may be written as[2]

$$\dot{x}_1 = x_2 \tag{1a}$$

$$\dot{x}_2 = K_1 x_1 + 2\omega x_4 + \alpha \cos x_5 \tag{1b}$$

$$\dot{x}_3 = x_4 \tag{1c}$$

$$\dot{x}_4 = -2\omega x_2 + K_2 x_3 + \alpha \sin x_5 \tag{1d}$$

$$\dot{x}_5 = u \tag{1e}$$

where x_{2k-1}, $x_{2k}(k = 1, 2)$ are the coordinates of the vehicle and, respectively, the components of the velocity; x_5 is the angle of orientation of the propulsion force with respect to a fixed direction in the present intertial system; α is the modulus of the propulsion force. As the angular velocity u of the propulsion vector is bounded, we have

$$|u| \le \Omega_{\max} \tag{2}$$

III. Optimum Problem

Let us write the controlled system (1) in the form

$$\phi_i \equiv \dot{x}_i - f_i(x_1, \dots, x_5, u) = 0 \qquad (i = 1, \dots, 5) \tag{3}$$

where $x_i \epsilon X$ and $u \epsilon U$. The functions

$$f_i(x_i, u) \qquad \frac{\partial o f_i(x_i, u)}{\partial x_i} \qquad (i, j = 1, \dots, 5) \tag{4}$$

are defined and continuous as $X \times U$. Consider two manifolds $S_1(t)$ and $S_F(t)$ defined by the equations

$$S_1 = \{x_i(0) = 0, \qquad i = 1, \dots, 5\}$$

$$S_F = \{x_2(T) = c; \qquad x_4(T) = d; \qquad x_5(T) = \varphi\} \tag{5}$$

Among the admissible controls $u \epsilon U = \{u(t), \ |u(t)| \le \Omega, \ 0 \le t \le T\}$ that transfer the representative point from the position $x_2 \epsilon S_1(t_2)$ at position $x_F \epsilon S_F(t_F)$, one has to determine those that minimize the functional

$$J = \int_0^T \mathrm{d}t \tag{6}$$

IV. Matching Conditions and Determination of Extremals

The Hamiltonian may be written as

$$H(x, u, \psi) = -1 + \sum_{i=1}^{5} \psi_i \dot{x}_i \tag{7}$$

and let

$$M(x, \psi) = \max_{u \epsilon U} H(x, p, u) \tag{8}$$

We note that $H = H_{\max}$ with respect to u if

$$u = \Omega_{\max} = \Omega \qquad \text{for} \qquad \psi_5 > 0 \tag{9}$$

The adjoint system

$$\dot{\psi}_i = -\frac{\partial H}{\partial x_i} \qquad (i = 1, \dots, 5) \tag{10}$$

is written as

$$\dot{\psi}_1 = \psi_2 K_1 \tag{11a}$$

$$\dot{\psi}_2 = \psi_1 + 2\omega\psi_4 \tag{11b}$$

$$\dot{\psi}_3 = -K_2 \psi_4 \tag{11c}$$

$$\dot{\psi}_4 = 2\omega\psi_2 - \psi_3 \tag{11d}$$

$$\dot{\psi}_5 = \alpha(-\psi_2 \sin x_5 + \psi_4 \cos x_5) \tag{11e}$$

The characteristic equation attached to the differential

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